ON THE DISCRETE SPECTRUM OF A FAMILY OF DIFFERENTIAL OPERATORS

MICHAEL SOLOMYAK

Dedicated to Victor Borisovich Lidskii on the occasion of his 80-th birthday

Abstract. A family $\mathbf{A}_\alpha$ of differential operators depending on a real parameter α is considered. The problem can be formulated in the language of perturbation theory of quadratic forms. The perturbation is only relatively bounded but not relatively compact with respect to the unperturbed form.

The spectral properties of the operator $\mathbf{A}_\alpha$ strongly depend on α. In particular, for $\alpha < \sqrt{2}$ the spectrum of $\mathbf{A}_\alpha$ below 1/2 is finite, while for $\alpha > \sqrt{2}$ the operator has no eigenvalues at all. We study the asymptotic behaviour of the number of eigenvalues as $\alpha \nearrow \sqrt{2}$. We reduce this problem to the one on the spectral asymptotics for a certain Jacobi matrix.

1. Introduction

In this paper we study the discrete spectrum of a family $\mathbf{A}_\alpha$ of differential operators in the space $\mathsf{L}^2(\mathbb{R}^2)$, defined by the differential expression

$$\mathcal{A}U = -U''_{xx} + \frac{1}{2}\big(-U''_{yy} + y^2 U\big) \tag{1.1}$$

and the "transmission condition" on the line $x = 0$:

$$U'_x(+0, y) - U'_x(-0, y) = \alpha y\, U(0, y), \qquad y \in \mathbb{R}. \tag{1.2}$$

In (1.2) α is a real parameter (the coupling constant). So, the differential expression which defines the action of the operator does not involve α. The parameter appears only in the condition (1.2) which defines the operator domain of $\mathbf{A}_\alpha$. The replacement $\alpha \mapsto -\alpha$ corresponds to the change of variables $y \mapsto -y$ which does not affect the spectrum. For this reason, below we discuss only $\alpha > 0$.

As we shall see, the spectrum $\sigma(\mathbf{A}_\alpha)$ of $\mathbf{A}_\alpha$ has the discrete component only for $\alpha < \sqrt{2}$, and we study its behaviour as $\alpha \nearrow \sqrt{2}$.

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Operators, containing the family $\mathbf{A}_\alpha$ as a special case, were suggested by U. Smilansky [8] as a model of an irreversible quantum system. Some important conclusions on the spectrum of $\mathbf{A}_\alpha$ were made in [8] "on the physical level of rigour". The first mathematical results on the subject were obtained in [9]. In [7] they are considerably extended. The work on [7] is yet unfinished and its results are not used in the present paper. However, below we mention some of these results in the course of our general discussion.

It was shown in [9] for $\alpha \neq \sqrt{2}$ and in [7] for $\alpha = \sqrt{2}$, that for any $\alpha \in \mathbb{R}$ the operator $\mathbf{A}_\alpha$, defined originally on the set of all functions from the Schwartz class, satisfying the condition (1.2), admits a unique self-adjoint realization. The special role of the value $\alpha = \sqrt{2}$ will be explained later. Below we refer to the values $\alpha < \sqrt{2}$ as *small* and to the values $\alpha > \sqrt{2}$ as *large*. The spectral properties of the operator $\mathbf{A}_\alpha$ for the small and the large values of the parameter α are quite different.

It is useful to consider the quadratic form $\mathbf{a}_\alpha[U]$ which formally corresponds to the operator $\mathbf{A}_\alpha$. It can be written as

$$\mathbf{a}_\alpha[U] = \mathbf{a}_0[U] + \alpha\mathbf{b}[U] \tag{1.3}$$

where

$$\mathbf{a}_0[U] = \int_{\mathbb{R}^2} \big(|U'_x|^2 + \frac{1}{2}(|U'_y|^2 + y^2|U|^2)\big)dxdy; \tag{1.4}$$

$$\mathbf{b}[U] = \int_{\mathbb{R}} y|U(0,y)|^2 dy. \tag{1.5}$$

We view $\mathbf{a}_0[U]$ as the unperturbed quadratic form and $\alpha\mathbf{b}[U]$ as the perturbation. An important feature of the problem studied stems from the fact that $\mathbf{b}[U]$ is only relatively bounded but not relatively compact with respect to the quadratic form $\mathbf{a}_0[U]$. For this reason, the standard results of the perturbation theory do not apply, which makes the study of the operators $\mathbf{A}_\alpha$ an interesting and non-trivial problem.

It turns out that the $\mathbf{a}_0$-bound of the quadratic form $\mathbf{b}[U]$ is exactly $1/\sqrt{2}$. This explains the role of the borderline value $\alpha = \sqrt{2}$. The techniques of quadratic forms does not apply to the large values of α. It was proved in [7] (and partly already in [9]) that the spectrum $\sigma(\mathbf{A}_\alpha)$ for $\alpha > \sqrt{2}$ is purely continuous and coincides with the whole of $\mathbb{R}$. The spectrum of the operator $\mathbf{A}_{\sqrt{2}}$ is also purely continuous and coincides with the half-line $[0, \infty)$.

The operator $\mathbf{A}_0$ can be easily studied by separation of variables. It expands into the orthogonal sum of the operators in $\mathsf{L}^2(\mathbb{R})$ given by

$$\mathbf{H}_n = -d^2/dx^2 + (n+1/2), \qquad n \in \mathbb{N}_0 := \{0, 1, \dots\}. \tag{1.6}$$

It follows that the spectrum $\sigma(\mathbf{A}_0)$ is absolutely continuous and coincides with the half-line $[1/2, \infty)$. Its multiplicity is n on each interval $(n-1/2, n+1/2)$, $n \in \mathbb{N}$.

The following statement, which describes the spectral properties of the operator A_α for α small, is a particular case of Theorem 6.2 in [9].

Proposition 1.1. *Let $\alpha < \sqrt{2}$. Then $\sigma_{\mathrm{ess}}(\mathbf{A}_\alpha) = \sigma(\mathbf{A}_0) = [1/2, \infty)$. The spectrum of $\mathbf{A}_\alpha$ below the threshold $\lambda_0 = 1/2$ lies in the interval $(0, 1/2)$, is always non-empty and consists of a finite number of eigenvalues.*

This structure of the lower spectrum is typical for relatively compact perturbations, a property which is violated in our case. Here another mechanism is in effect and leads to the same result, but only for small values of α. We discuss this mechanism in the final section 5.

Given a self-adjoint operator $\mathbf{T}$ in a Hilbert space $\mathfrak{H}$ and a real number s, we denote

$$N_+(s; \mathbf{T}) = \dim E^{\mathbf{T}}(s, \infty)\mathfrak{H}, \qquad N_-(s; \mathbf{T}) = \dim E^{\mathbf{T}}(-\infty, s)\mathfrak{H}$$

where $E^{\mathbf{T}}(\cdot)$ is the spectral measure of $\mathbf{T}$. If, say, $N_-(s; \mathbf{T}) < \infty$, then the spectrum of the operator $\mathbf{T}$ on the interval $(-\infty, s)$ reduces to a finite number of eigenvalues (counting their multiplicities), and $N_-(s; \mathbf{T})$ is equal to this number.

Our goal in this paper is study of the function $N_-(1/2; \mathbf{A}_\alpha)$ as $\alpha \nearrow \sqrt{2}$. We shall see that $N_-(1/2; \mathbf{A}_\alpha) \to \infty$, and calculate the asymptotics of this function. This complements Proposition 1.1 by giving a quantitative characteristic of the discrete part of $\sigma(\mathbf{A}_\alpha)$. Probably, Theorem 3.1 should be considered as the central result of the paper. It establishes the equality $N_-(1/2 - \varepsilon; \mathbf{A}_\alpha) = N_+(\sqrt{2}/\alpha; \mathbf{J}(\varepsilon))$ where $\varepsilon \in (0, 1/2)$ and $\mathbf{J}(\varepsilon)$ is a certain Jacobi matrix. For the operator family $\mathbf{A}_\alpha$ this is an analog of the classical Birman – Schwinger principle.

2. Quadratic form $\mathbf{a}_\alpha$

The quadratic form $\mathbf{a}_0$ given by (1.4) is positive definite and closed on the natural form-domain

$$D := \operatorname{Dom} \mathbf{a}_0 = \{U \in \mathsf{H}^1(\mathbb{R}^2) : \mathbf{a}_0[U] < \infty\}$$

where, as usual, H^1 stands for the Sobolev space. The self-adjoint operator in $\mathsf{L}^2(\mathbb{R}^2)$, generated by the quadratic form $\mathbf{a}_0[U]$, is $\mathbf{A}_0$, i.e. the operator (1.1) – (1.2) for $\alpha = 0$.

It is convenient to express both quadratic forms $\mathbf{a}_0$ and $\mathbf{b}$ in terms of the decomposition of U into the series in the (normalized in $\mathsf{L}^2(\mathbb{R})$) Hermite functions in y:

$$U(x,y) = \sum_{n\in\mathbb{N}_0} u_n(x)\chi_n(y). \tag{2.1}$$

We often identify a function $U(x,y)$ with the sequence $\{u_n(x)\}$ and write $U \sim \{u_n\}$. This identification is a unitary mapping of the space $\mathsf{L}^2(\mathbb{R}^2)$ onto the Hilbert space $\ell^2(\mathbb{N}_0, \mathsf{L}^2(\mathbb{R}))$. Let us recall the recurrence relation for the functions χ_n:

$$\sqrt{n+1}\chi_{n+1}(y) - \sqrt{2}y\chi_n(y) + \sqrt{n}\chi_{n-1}(y) = 0, \qquad n \in \mathbb{N}_0. \tag{2.2}$$

Substituting in (1.4) the representation (2.1) of the function U, we find

$$\mathbf{a}_0[U] = \sum_{n\in\mathbb{N}_0} \mathbf{h}_n[u_n] =: \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}} \big(|u_n'|^2 + (n+1/2)|u_n|^2\big)dx. \tag{2.3}$$

The equality (2.3) shows that the decomposition (2.1) diagonalyzes the quadratic form $\mathbf{a}_0[U]$ and hence, reduces the operator $\mathbf{A}_0$. This immediately implies the decomposition of $\mathbf{A}_0$ into the orthogonal sum of the operators $\mathbf{H}_n$, see (1.6), and hence the structure of the spectrum $\sigma(\mathbf{A}_0)$, described in the Introduction.

In the same way, taking (2.2) into account, we find that

$$\mathbf{b}[U] = \sum_{n\in\mathbb{N}_0} \sqrt{2n}\,\mathrm{Re}\big(u_n(0)\overline{u_{n-1}(0)}\big). \tag{2.4}$$

Lemma 2.1. *(cf.* [9], *section 6). The quadratic form* $\mathbf{b}[U]$ *is well-defined on* D*, and*

$$\sqrt{2}\,|\mathbf{b}[U]| \le \mathbf{a}_0[U], \qquad \forall U \in D. \tag{2.5}$$

Proof. Our argument is based upon the inequality

$$2\gamma|u(0)|^2 \le \int_{\mathbb{R}} \big(|u'|^2 + \gamma^2|u|^2\big)dx, \qquad \forall u \in \mathsf{H}^1(\mathbb{R}),\ \gamma > 0. \tag{2.6}$$

Its proof is elementary and we skip it. It is also easy to show that the equality in (2.6) is attained on the one-dimensional subspace in $\mathsf{H}^1(\mathbb{R})$, generated by the function

$$\widetilde{u}_\gamma(x) := (2\gamma)^{-1/2}e^{-\gamma|x|}. \tag{2.7}$$

Here the factor $(2\gamma)^{-1/2}$ is chosen in such a way that

$$\int_{\mathbb{R}} \big(|\widetilde{u}_\gamma'|^2 + \gamma^2 |\widetilde{u}_\gamma|^2\big) dx = 1.$$

We derive from (2.4):

$$\sqrt{2}|\mathbf{b}[U]| \le \sum_{n\in\mathbb{N}} \sqrt{n}(|u_n(0)|^2 + |u_{n-1}(0)|^2)$$
$$= \sum_{n\in\mathbb{N}_0} \big(\sqrt{n} + \sqrt{n+1}\big)|u_n(0)|^2.$$

Since $\sqrt{n} + \sqrt{n+1} < \sqrt{2(2n+1)}$, we conclude from (2.6) that

$$\sqrt{2}|\mathbf{b}[U]| \le \sum_{n\in\mathbb{N}_0} \mathbf{h}_n[u_n] = \mathbf{a}_0[U], \qquad \forall U \sim \{u_n\} \in D, \tag{2.8}$$

whence (2.5). □

It follows from Lemma 2.1 that for $0 < \alpha < \sqrt{2}$ the quadratic form $\mathbf{a}_\alpha[U]$ is positive definite:

$$\mathbf{a}_\alpha[U] \ge \big(1 - \frac{\alpha}{\sqrt{2}}\big)\mathbf{a}_0[U] \ge \frac{1}{2}\big(1 - \frac{\alpha}{\sqrt{2}}\big)\|U\|^2, \qquad U \in D \tag{2.9}$$

(here and in the sequel $\|U\| := \|U\|_{\mathsf{L}^2(\mathbb{R}^2)}$). It is also closed, cf. e.g. [2], Lemma 1.1. The operator $\mathbf{A}_\alpha$ for such α can be defined as the self-adjoint operator in $\mathsf{L}^2(\mathbb{R}^2)$, associated with the quadratic form $\mathbf{a}_\alpha[U]$.

3. Function $N_-(\frac{1}{2} - \varepsilon; \mathbf{A}_\alpha)$

We are interested in the lower spectrum of the operator $\mathbf{A}_\alpha$, i.e. in the part of spectrum lying below the point $1/2 = \inf \sigma(\mathbf{A}_0)$. By (2.9), this part of $\sigma(\mathbf{A}_\alpha)$ lies in the interval $1 - \alpha/\sqrt{2} \le 2\lambda < 1$. The general perturbation theory gives no further information, since the quadratic form $\mathbf{b}$ is only relatively bounded but not relatively compact with respect to $\mathbf{a}_0$. However, we reduce the problem to a simpler one, for a certain Jacobi operator in $\ell^2(\mathbb{N}_0)$. This reduction allows us to handle the original problem.

Fix a number ε, $0 < \varepsilon < 1/2$ and consider a zero-diagonal Jacobi matrix $\mathbf{J}(\varepsilon)$ with the entries

$$j_{n,n-1}(\varepsilon) = j_{n-1,n}(\varepsilon) = \frac{n^{1/2}}{2(n+\varepsilon)^{1/4}(n-1+\varepsilon)^{1/4}}, \qquad n \in \mathbb{N}.$$

All the other entries of the matrix are equal to zero. We use the same symbol $\mathbf{J}(\varepsilon)$ for the operator in $\ell^2(\mathbb{N}_0)$, generated by this matrix. The operator $\mathbf{J}(\varepsilon)$ is bounded and self-adjoint, its spectrum is invariant

under the reflection $\lambda \mapsto -\lambda$. It is well known that $\sigma_{\mathrm{ac}}(\mathbf{J}(\varepsilon)) = [-1,1]$. Besides, the operator may have (and actually, has) simple eigenvalues $\pm\lambda_n, \lambda_n > 1$, with the only possible accumulation points at $\lambda = \pm 1$.

Theorem 3.1. *For any $\alpha \in (0, \sqrt{2})$, define $s(\alpha) = \sqrt{2}/\alpha$. Then for an arbitrary $\varepsilon \in (0, 1/2)$ the equality is satisfied:*

$$N_-(1/2 - \varepsilon; \mathbf{A}_\alpha) = N_+(s(\alpha), \mathbf{J}(\varepsilon)) = N_-(-s(\alpha), \mathbf{J}(\varepsilon)). \tag{3.1}$$

The equality (3.1) can be considered as one more manifestation of the general Birman – Schwinger principle.

Proof. According to the variational principle,

$$N_-(1/2 - \varepsilon; \mathbf{A}_\alpha) = \max_{\mathcal{F} \in \mathfrak{F}(\varepsilon)} \dim \mathcal{F} \tag{3.2}$$

where $\mathfrak{F}(\varepsilon)$ is the set of all subspaces $\mathcal{F} \subset D$, such that

$$\mathbf{a}_\alpha[U] - (1/2 - \varepsilon)\|U\|^2_{\mathsf{L}^2(\mathbb{R}^2)} < 0, \qquad \forall U \in \mathcal{F},\ U \neq 0. \tag{3.3}$$

Set

$$\|U\|^2_\varepsilon = \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}} \big(|u_n'|^2 + (n+\varepsilon)|u_n|^2\big)dx, \qquad U \sim \{u_n\}.$$

For any $\varepsilon > 0$ and any $U \in \mathfrak{H}$ the quadratic form $\|U\|^2_\varepsilon$ can be estimated through $\mathbf{a}_0[U]$ from above and from below and hence, can be taken as a metric form on D. The inequality (3.3) can be re-written as

$$\|U\|^2_\varepsilon + \alpha \sum_{n \in \mathbb{N}} \sqrt{2n}\, \mathrm{Re}\big(u_n(0)\overline{u_{n-1}(0)}\big) < 0. \tag{3.4}$$

Consider the subspace $\widetilde{D}(\varepsilon)$ in D, formed by the elements

$$\widetilde{U} \sim \{C_n \widetilde{u}_{\sqrt{n+\varepsilon}}\},\ \{C_n\} \in \ell^2(\mathbb{N}_0),$$

where the elements $\widetilde{u}_\gamma$ for any $\gamma > 0$ are given by (2.7). Note that $\|\widetilde{U}\|_\varepsilon = \|\{C_n\}\|_{\ell^2}$. Let Π_ε stand for the operator which projects D onto $\widetilde{D}(\varepsilon)$ and is orthogonal in the metric $\|\cdot\|_\varepsilon$. If $U \sim \{u_n\} \in D$, then

$$\widetilde{U}_\varepsilon := \Pi_\varepsilon U \sim \{C_n \widetilde{u}_{\sqrt{n+\varepsilon}}\}$$

where

$$C_n = \int_{\mathbb{R}} \big(u_n' \widetilde{u}'_{\sqrt{n+\varepsilon}} + (n+\varepsilon) u_n \widetilde{u}_{\sqrt{n+\varepsilon}}\big)dx = 2^{1/2}(n+\varepsilon)^{1/4} u_n(0).$$

We see that

$$C_n \widetilde{u}_{\sqrt{n+\varepsilon}}(0) = u_n(0), \qquad \forall n \in \mathbb{N}_0. \tag{3.5}$$

If in the inequality (3.4) we replace U by $\widetilde{U}_\varepsilon$, the first term in the left-hand side does not increase and the second remains unchanged, so

that the inequality remains valid. In other words, if a subspace $\mathcal{F} \subset D$ belongs to the class $\mathfrak{F}(\varepsilon)$, then also $\Pi_\varepsilon \mathcal{F} \in \mathfrak{F}(\varepsilon)$.

On the other hand, assume that $\mathcal{F}, \mathcal{F}'$ are two subspaces of the class $\mathfrak{F}(\varepsilon)$, such that $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{F} \subset \widetilde{D}(\varepsilon)$. Suppose that there exists an element $U \sim \{u_n\} \in \mathcal{F}'$ orthogonal to $\mathcal{F}$ in ε-metric. Then by (3.5) $u_n(0) = 0$ for all $n \in \mathbb{N}_0$. This yields $\mathbf{b}[U] = 0$, which contradicts (3.4). This means that our assumption implies $\mathcal{F} = \mathcal{F}'$.

It follows from these remarks that along with (3.2) the next equality holds:

$$N_-(1/2 - \varepsilon; \mathbf{A}_\alpha) = \max_{\mathcal{F} \in \mathfrak{F}(\varepsilon),\ \mathcal{F} \subset \widetilde{D}(\varepsilon)} \dim \mathcal{F}.$$

For any $\widetilde{U} \sim \{C_n \widetilde{u}_{\sqrt{n+\varepsilon}}\} \in \widetilde{D}(\varepsilon)$ we have

$$\begin{aligned} \|\widetilde{U}\|_\varepsilon^2 + \alpha \mathbf{b}[\widetilde{U}] &= \sum_{n \in \mathbb{N}_0} |C_n|^2 + 2s^{-1} \sum_{n \in \mathbb{N}} j_{n,n-1}(\varepsilon) \operatorname{Re}(C_n \overline{C_{n-1}}) \\ &= \|g\|_{\ell^2}^2 + s^{-1} \big(\mathbf{J}(\varepsilon) g, g\big)_{\ell^2}, \qquad g = \{C_n\} \in \ell^2. \end{aligned}$$

The sum in the right-hand side is the quadratic form of the operator $\mathbf{I} + s^{-1}\mathbf{J}(\varepsilon)$. Now (3.1) is implied by the variational principle and the symmetry of $\sigma(\mathbf{J}(\varepsilon))$. $\square$

Theorem 3.1 does not apply to the most interesting case $\varepsilon = 0$, since $j_{1,0}(0) = \infty$. However, we can restrict the quadratic form $\big(\mathbf{J}(\varepsilon)g, g\big)_{\ell^2}$ to the subspace $\big\{g = \{C_n\} : \ C_0 = 0\big\}$ of codimension 1. This may shift the number of eigenvalues no more than by one. For the problem obtained, the passage to the limit as $\varepsilon \to 0$ is already possible, and the resulting zero-diagonal Jacobi matrix is $\mathbf{J}_0$ whose off-diagonal entries are given by

$$2j_{n,n-1} = 2j_{n-1,n} = (1 - n^{-1})^{-1/4}, \qquad n - 1 \in \mathbb{N}. \tag{3.6}$$

So, we arrive at the following result.

Theorem 3.2. *Let $\alpha \in (0, \sqrt{2})$ and $s(\alpha) = \sqrt{2}/\alpha$. Then either $N_-(1/2; \mathbf{A}_\alpha) = N_+(s; \mathbf{J}_0)$, or $N_-(1/2; \mathbf{A}_\alpha) = N_+(s; \mathbf{J}_0) + 1$.*

Theorem 3.2 reduces the problem of the asymptotic behaviour of the function $N_-(1/2; \mathbf{A}_\alpha)$ as $\alpha \nearrow \sqrt{2}$ to the question about the asymptotics of the eigenvalues of the matrix $\mathbf{J}_0$, lying above the point $\lambda = 1$. We could not find the corresponding result in the literature, so that we derive it the next section. Here is the formulation.

Theorem 3.3. *Let $\mathbf{J}$ be a zero-diagonal Jacobi matrix with the off-diagonal entries*

$$j_{n,n-1} = j_{n-1,n} = 1/2 + qn^{-1}(1 + o(1)) \tag{3.7}$$

where $q = const,\ q > 0$. Then the operator $\mathbf{J}$ has the infinite number of non-degenerate eigenvalues $\pm\lambda_k(\mathbf{J})$, such that

$$\lambda_k(\mathbf{J}) = 1 + \frac{2q^2}{k^2}(1 + o(1)), \qquad k \to \infty. \tag{3.8}$$

These eigenvalues exhaust the spectrum of $\mathbf{J}$ outside $[-1, 1]$. Equivalently to (3.8),

$$N_+(s; \mathbf{J}) \sim \frac{q\sqrt{2}}{\sqrt{s-1}}, \qquad s \searrow 1. \tag{3.9}$$

It follows from (3.6) that the entries of the matrix $\mathbf{J}_0$ satisfy (3.7) with $q = 1/8$. Therefore, Theorems 3.2 and 3.3 (equality (3.9)) immediately imply the asymptotic formula

$$N_-(1/2; \mathbf{A}_\alpha) \sim \frac{1}{4\sqrt{2(s(\alpha) - 1)}}, \qquad s(\alpha) = \sqrt{2}/\alpha,\ \alpha \searrow \sqrt{2}. \tag{3.10}$$

4. Proof of Theorem 3.3

The main ingredient of the proof is a result by W. Van Asshe [1] on a class of the orthogonal polynomials on the real axis, namely of the so-called Pollaczek polynomials $P^\lambda(x; a, b)$, see e.g. [3]. They depend on three real parameters λ, a, b but we need only their particular case for $b = 0,\ a = -r < 0$ and $\lambda > r$. The monic Pollaczek polynomials $Q^\lambda(x; r)$, i.e. polynomials $P^\lambda(x; -r, 0)$, normalized in such a way that their leading coefficient becomes 1, satisfy the recurrent relation

$$Q^\lambda_{n+1}(x; r) = xQ^\lambda_n(x; r) - p_n(\lambda, r)Q^\lambda_{n-1}(x; r),$$
$$p_n(\lambda, r) = \frac{n(n + 2\lambda - 1)}{4(n - r + \lambda - 1)(n - r + \lambda)}, \qquad n \in \mathbb{N}.$$

The polynomials Q^λ_n correspond to the zero-diagonal Jacobi matrix $\mathbf{J}(\lambda, r)$ whose off-diagonal entries are

$$j_{n,n-1} = j_{n-1,n} = \sqrt{p_n(\lambda, r)}. \tag{4.1}$$

It was proven in [1], Section III that the spectrum of $\mathbf{J}(\lambda, r)$, lying outside the segment $[-1, 1]$, consists of the non-degenerate eigenvalues $\pm\mu_k = \pm\mu_k(\lambda, r)$ where μ_k satisfy the equation

$$\lambda - \frac{r\mu}{\sqrt{\mu^2 - 1}} = -k, \qquad k \in \mathbb{N}_0.$$

This gives

$$\mu_k = \left(1 - \frac{r^2}{(k + \lambda)^2}\right)^{-1/2} = 1 + \frac{r^2}{2k^2} + o\big(\frac{1}{k^2}\big), \qquad k \to \infty. \tag{4.2}$$

Another ingredient is a variational principle for the eigenvalues of Jacobi matrices, see [6], Lemma III.1. This variational principle is an almost immediate consequence of Sturm's comparison theorem, see e.g. [5], Theorem 1 on p. 152. Below we present its formulation for a particular case we need in this paper.

Lemma 4.1. *Let* $\mathbf{J}$, $\mathbf{J}'$ *be Jacobi matrices with the zero diagonal entries and the off-diagonal entries* $j_{n,n-1} = 1/2 + b_n$, $j'_{n,n-1} = 1/2 + b'_n$, *such that* $0 \le b_n \le b'_n$ *for all* n *and* $b'_n \to 0$. *Then* $\sigma_{ess}(\mathbf{J}) = \sigma_{ess}(\mathbf{J}') = [-1,1]$ *and for any* $s > 1$

$$N^+(s;\mathbf{J}) \le N^+(s;\mathbf{J}').$$

Now we are in a position to prove Theorem 3.3. It follows from Lemma 4.1 that the asymptotic behaviour of the eigenvalues does not depend on the term $o(1)$ in (3.7). The entries $j_{n,n-1}$ in (4.1) satisfy $2j_{n,n-1} \sim 1 + r/n$. Clearly, (3.8) is a direct consequence of (4.2).

5. Concluding remarks

5.1. Here we explain, why for $\alpha < \sqrt{2}$ the structure of the spectrum $\sigma(\mathbf{A}_\alpha)$ of the operator $\mathbf{A}_\alpha$ below the threshold $1/2$ is the same as if the perturbation were relatively compact. Of course, one such explanation is given by the proof of Theorem 3.1, however we shall present also one more argument, of a somewhat more heuristic nature. A rigorous version of this argument was used in [9] for the proof of Theorem 6.2.

The quadratic form $\mathbf{b}[U]$, see (1.5) and (2.4), is the sum of terms

$$\mathbf{b}_n[U] = \sqrt{2n}\,\mathrm{Re}\big(u_n(0)\overline{u_{n-1}(0)}\big),$$

each of rank two. The quadratic form $\mathbf{b}_n[U]$ interacts only with the terms $\mathbf{h}_{n-1}[u_{n-1}]$ and $\mathbf{h}_n[u_n]$ in the representation (2.3) of the quadratic form $\mathbf{a}_0[U]$. The term $\mathbf{h}_n[u_n]$ corresponds to the operator $\mathbf{H}_n$, see (1.6), whose spectrum is $[n+1/2,\infty)$. The perturbation of the spectrum, brought by the term $\alpha\mathbf{b}_n[U]$, does not reach the point $\lambda_0 = 1/2$, provided that $\alpha < \sqrt{2}$ and n is large enough. This means that effectively we are dealing with a finite rank perturbation, as soon as we restrict ourselves with the small values of the coupling parameter and are interested only in the lower part of $\sigma(\mathbf{A}_\alpha)$.

5.2. In the paper [9] the operator family $\mathbf{A}_\alpha$ was considered in a more general setting. Namely the operators act in the space $\mathsf{L}^2(\Gamma\times\mathbb{R})$ where Γ is a *metric star graph*, i.e. a graph with m bonds $\mathcal{B}_1,\dots,\mathcal{B}_m$, $1 \le m < \infty$, all emanating from a common vertex o. Let us recall that each bond of a metric graph is viewed as a line segment of finite or infinite

length. The real axis $\mathbb{R}$ can be considered as the star graph with two bonds (so that $m = 2$), each of infinite length, and with $o = 0$.

Let us identify each bond $\mathcal{B}_j$ with the segment $[0, B_j)$, where $B_j \le \infty$ is the length of $\mathcal{B}_j$. We denote by x the coordinate along each bond (dropping the index j); the value $x = 0$ corresponds to the point o.

The action of the operator $\mathbf{A}_\alpha$ in this, more general case is defined by the same equality (1.1), in which y denotes the coordinate along the additional straight line. The condition (1.2) is replaced by the matching conditions

$$U^1(0,y) = \ldots = U^m(0,y);$$
$$U^1_x(0,y) + \ldots + U^m_x(0,y) = \alpha y U(0,y)$$

where U^j stands for the restriction of U to the bond $\mathcal{B}_j$. Besides, the Dirichlet condition $U^j(B_j, y) = 0$ is imposed for each bond of finite length.

Theorem 6.2 in [9] (cf. Proposition 1.1 of the present paper) was proved for this general version of the operator $\mathbf{A}_\alpha$. The only difference with the particular case $\Gamma = \mathbb{R}$ is that the $\mathbf{a}_0$-bound of the quadratic form $\mathbf{b}$ is $\sqrt{2}/m$, cf. (2.5). Correspondingly, the techniques developed in the present paper allows one to prove an analog of the asymptotic relation (3.10). The only distinction is that for any star graph with m bonds we have to take $\alpha \nearrow m/\sqrt{2}$ and $s(\alpha) = m/(\alpha\sqrt{2})$.

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Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: michail.solomyak@weizmann.ac.il